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Operator inequalities associated with $A \log A$ via Specht ratio

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Dedicated to the memory of Professor Shōzo Koshi in deep sorrow

Abstract

An operator means a bounded linear operator on a Hilbert space H. We obtained the basic property between Specht ratio $S(1)$ and generalized Kantorovich constant $K(p)$ in [Math. Inequal. Appl., in press], that is, Specht ratio $S(1)$ can be expressed by generalized Kantorovich constant $K(p)$: $S(1) = \mathrm{e}^{K'(1)}$. We shall investigate several product type and difference type inequalities associated with $A \log A$ by applying this basic property to several Kantorovich type inequalities.

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Keywords: Kantorovich inequality; Specht ratio

1. Introduction

An operator A is said to be positive operator (denoted by $T \geqslant 0$) if $(Ax, x) \geqslant 0$ for all x in H and also A is said to be strictly positive operator (denoted by $A > 0$) if A is invertible positive operator.

Definition 1. Let $h > 1$. $S(h, p)$ is defined by

$$S(h, p) = \frac{h^{\frac{p}{h^p - 1}}}{\mathrm{e} \log h^{\frac{p}{h^p - 1}}} \quad \text{for any real number } p \tag{1.1}$$

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and $S(h, p)$ is denoted by $S(p)$ briefly. Especially $S(1) = S(h, 1) = \frac{h^{\frac{1}{h-1}}}{\mathrm{e}\log h^{\frac{1}{h-1}}}$ is said to be *Specht ratio* and $S(1) > 1$ is well known.

Let $h > 1$. *The generalized Kantorovich constant* $K(h, p)$ is defined by

$$K(h, p) = \frac{(h^p - h)}{(p-1)(h-1)}\left(\frac{(p-1)}{p}\frac{(h^p - 1)}{(h^p - h)}\right)^p \quad \text{for any real number } p \tag{1.2}$$

and $K(h, p)$ is denoted by $K(p)$ briefly.

Basic property [13]. *The following basic property among* $S(1)$, $K'(1)$ *and* $K'(0)$ *holds*:

$$S(1) = \mathrm{e}^{K'(1)} = \mathrm{e}^{-K'(0)}$$
$$\left(\text{i.e., } S(1) = \exp\left[\lim_{p\to 1} K'(p)\right] = \exp\left[-\lim_{p\to 0} K'(p)\right]\right), \tag{1.3}$$

$$K(0) = K(1) = 1 \quad \left(\text{i.e., } \lim_{p\to 0} K(p) = \lim_{p\to 1} K(p) = 1\right), \tag{1.4}$$

$$S(1) = \lim_{p\to 1} K(p)^{\frac{1}{p-1}} = \lim_{p\to 0} K(p)^{\frac{-1}{p}}. \tag{1.5}$$

Refer to Fig. 1 for the relation between $K(p)$ and $S(p)$. The relation (1.3) is quite important in this paper, so we state its proof for the sake of convenience. In fact $K'(p)$ can be written as follows:

$$K'(p) = \frac{\left(\frac{(p-1)}{p}\frac{(h^p-1)}{(h^p-h)}\right)^p}{(h-1)(h^p-1)} \times \left\{\frac{h^p(h^p-1+p-hp)\log h + (h^p-1)(h^p-h)\log\frac{(p-1)(h^p-1)}{p(h^p-h)}}{p-1}\right\}. \tag{$*$}$$

By using L'Hospital theorem to $(*)$, we have

$$\begin{aligned}\lim_{p\to 1} K'(p) &= \frac{h-1}{h\log h}\frac{1}{(h-1)^2} \\ &\quad\times \left\{h\log h(h\log h + 1 - h) + (h-1)h\log h \log\left[\frac{h-1}{h\log h}\right]\right\} \\ &= \frac{h}{h-1}\log h - 1 + \log\left[\frac{h-1}{h\log h}\right]\end{aligned}$$

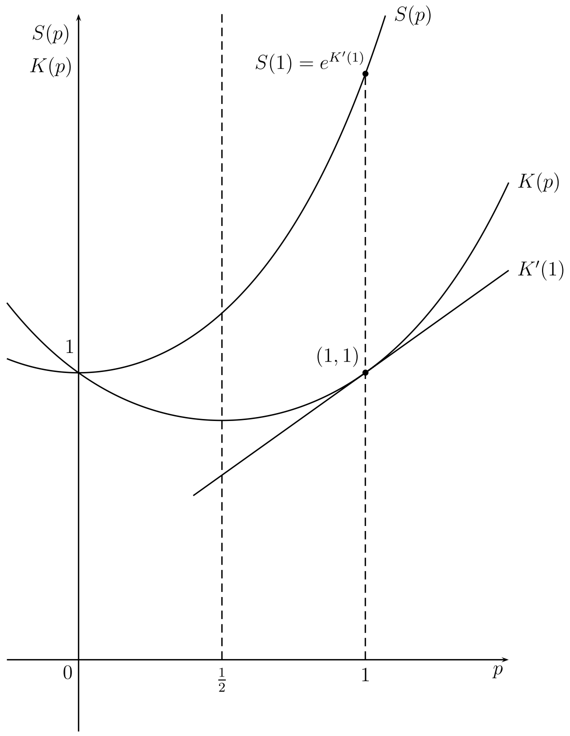

Fig. 1. Relation between $K(p)$ and $S(p)$.

$$= \log\left[\frac{h^{\frac{1}{h-1}}}{\mathrm{e}\log h^{\frac{1}{h-1}}}\right]$$
$$= \log S(1)$$

so that we have $S(1) = \mathrm{e}^{K'(1)}$ and also $S(1) = \mathrm{e}^{-K'(0)}$ by the same way. We remark that (1.5) is an immediate consequence of (1.3) by L'Hospital theorem. Another nice relation between $K(p)$ and $S(1)$ is in [26].

Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. The celebrated Kantorovich inequality asserts that

$$\frac{(1+h)^2}{4h}(Ax, x)^{-1} \geqslant (A^{-1}x, x) \geqslant (Ax, x)^{-1} \tag{1.6}$$

holds for every unit vector x and this inequality is just equivalent to the following one

$$\frac{(1+h)^2}{4h}(Ax, x)^{2} \geqslant (A^{2}x, x) \geqslant (Ax, x)^{2} \tag{1.7}$$

holds for every unit vector x. We remark that $K(h, p)$ in (1.2) is an extension of $\frac{(1+h)^2}{4h}$ in (1.6) and (1.7), in fact, $K(h, -1) = K(h, 2) = \frac{(1+h)^2}{4h}$ holds.

Many papers on Kantorovich inequality have been published. Among others, there is a long research series by Mond–Pečarić, we cite [21–23] for examples.

We state the Jensen inequality as follows. (cf. [1, Theorem 4;3,4;17, Theorem 2.1].)

Jensen inequality. *Let f be an operator concave function on an interval I. If Φ is normalized positive linear map, then*

$$f(\Phi(A)) \geqslant \Phi(f(A))$$

for every self adjoint operator A on a Hilbert space H whose spectrum is contained in I.

On the other hand, the relative operator entropy $S(X|Y)$ for $X > 0$ and $Y > 0$ is defined in [7] as an extension of the operator entropy $S(X|I) = -X \log X$

$$S(X|Y) = X^{\frac{1}{2}} \left[\log\left(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}}\right)\right] X^{\frac{1}{2}}. \tag{1.8}$$

By using this $S(X|Y)$, we define $T(X|Y)$ for $X > 0$ and $Y > 0$;

$$T(X|Y) = (X\sharp Y)X^{-1}S(X|Y)X^{-1}(X\sharp Y), \tag{1.9}$$

where $X\sharp Y = X^{\frac{1}{2}}(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}})^{\frac{1}{2}} X^{\frac{1}{2}}$. The power mean $X\sharp_p Y = X^{\frac{1}{2}}(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}})^p X^{\frac{1}{2}}$ for $p \in [0,1]$ is in [16] as an extension of $X\sharp Y$. We shall verify that $T(X|Y) = \lim_{p\to 1}(X\sharp_p Y)'$ in Proposition 3.2 and we remark that $S(X|Y) = \lim_{p\to 0}(X\sharp_p Y)'$ shown in [7].

In this paper $\lim_{p\to 0} F(p)$ means $\lim_{p\to +0} F(p)$ and also $\lim_{p\to 1} F(p)$ means $\lim_{p\to 1-0} F(p)$, incidentally $F'(0)$ means $F'(+0)$ and $F'(1)$ means $F'(1-0)$ and so on.

Next we state the following several Kantorovich type inequalities.

Theorem A. *Let A be strictly positive operator on a Hilbert space H satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and Φ be a normalized positive linear map on $B(H)$. Let $p \in (0,1)$. Then the following inequalities hold*:

(i) $\Phi(A)^p \geqslant \Phi(A^p) \geqslant K(p)\Phi(A)^p$,
(ii) $\Phi(A)^p \geqslant \Phi(A^p) \geqslant \Phi(A)^p - g(p)I$,

where $g(p) = m^p\left[\frac{h^p-h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\right]$ *and $K(p)$ is defined in* (1.2).

The right hand side inequalities of (i) and (ii) in Theorem A follow by [18, Corollary 2.6;23] and the left hand side one of (i) follows by Jensen inequality since $f(A) = A^p$ is operator concave for $p \in [0,1]$. More general forms than Theorem A are in [17] and related results to Theorem A are in [19,20].

Theorem B. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1I \geqslant A \geqslant m_1I > 0$ and $M_2I \geqslant B \geqslant m_2I > 0$. Put $m = m_1m_2$, $M = M_1M_2$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0, 1)$. Then the following inequalities hold*:

(i) $(A * B)^p \geqslant A^p * B^p \geqslant K(p)(A * B)^p$,
(ii) $(A * B)^p \geqslant A^p * B^p \geqslant (A * B)^p - g(p)I$,

where $g(p) = m^p \left[\frac{h^p-h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\right]$ *and $K(p)$ is defined in* (1.2).

The right hand side inequalities of (i) and (ii) follow by [25, Theorem 16] and the left hand side one of (i) follows by [10;25, Theorem 1].

Theorem C. *Let A, B, C and D be strictly positive operators on a Hilbert space H such that $M_1I \geqslant A \otimes B \geqslant m_1I > 0$ and $M_2I \geqslant C \otimes D \geqslant m_2I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0, 1)$. Then the following inequalities hold*:

(i) $(A * B)\sharp_p(C * D) \geqslant (A\sharp_pC) * (B\sharp_pD) \geqslant K(p)(A * B)\sharp_p(C * D)$,
(ii) $(A * B)\sharp_p(C * D) \geqslant (A\sharp_pC) * (B\sharp_pD) \geqslant (A * B)\sharp_p(C * D) - g(p)I(A * B)$,

where $g(p) = m^p \left[\frac{h^p-h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\right]$ *and $K(p)$ is defined in* (1.2).

The right hand side inequalities of (i) and (ii) follow by [18, Corollary 4.4] and the left hand side inequality of (i) follows by [2, Theorem 4.1] and also it follows by a corollary of [5, Theorem 5].

Theorem D. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1I \geqslant A \geqslant m_1I > 0$ and $M_2I \geqslant B \geqslant m_2I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$. Let $p \in (0, 1)$ and also let Φ be normalized positive linear map on $B(H)$. Then the following inequalities hold*:

(i) $\Phi(A)\sharp_p\Phi(B) \geqslant \Phi(A\sharp_pB) \geqslant K(p)\Phi(A)\sharp_p\Phi(B)$,
(ii) $\Phi(A)\sharp_p\Phi(B) \geqslant \Phi(A\sharp_pB) \geqslant \Phi(A)\sharp_p\Phi(B) - g(p)\Phi(A)$,

where $g(p) = m^p \left[\frac{h^p-h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\right]$ *and $K(p)$ is defined in* (1.2).

The right hand side inequalities of (i) and (ii) follow by [18, Corollary 3.5] and the left hand side one of (i) follows by [1,16].

The following result is contained in [18, Corollary 4.11] together with [5, Corollary 8].

Theorem E′. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geqslant A \geqslant m_1 I > 0$ and $M_2 I \geqslant B \geqslant m_2 I > 0$. Let $p \in (0,1)$ and also $m = m_1^{\frac{1}{p}} M_2^{\frac{-1}{1-p}}$, $M = M_1^{\frac{1}{p}} m_2^{\frac{-1}{1-p}}$ and $h = \frac{M}{m} = \left(\frac{M_1}{m_1}\right)^{\frac{1}{p}} \left(\frac{M_2}{m_2}\right)^{\frac{1}{1-p}} > 1$. Then the following inequalities hold*:

(i) $\left(A^{\frac{1}{p}} * I\right)^p \left(B^{\frac{1}{1-p}} * I\right)^{1-p} \geqslant A * B \geqslant K(p) \left(A^{\frac{1}{p}} * I\right)^p \left(B^{\frac{1}{1-p}} * I\right)^{1-p}$,

(ii) $\left(A^{\frac{1}{p}} * I\right)^p \left(B^{\frac{1}{1-p}} * I\right)^{1-p} \geqslant A * B \geqslant \left(A^{\frac{1}{p}} * I\right)^p \left(B^{\frac{1}{1-p}} * I\right)^{1-p} - g(p)(B * I)$,

where $g(p) = m^p \left[\frac{h^p - h}{h-1} + (1-p)\left(\frac{h^p - 1}{p(h-1)}\right)^{\frac{p}{p-1}}\right]$ *and $K(p)$ is defined in* (1.2).

In fact put $A_3 = A^p$ and $B_3 = B^{1-p}$, then $M_1^p I \geqslant A_3 \geqslant m_1^p I > 0$ and $M_2^{1-p} I \geqslant B_3 \geqslant m_2^{1-p} I > 0$ under the hypotheses of Theorem E. By applying Theorem E′ to A_3 and B_3, put $m_3 = m_1^{p\frac{1}{p}} M_2^{(1-p)\frac{-1}{1-p}} = \frac{m_1}{M_2}$, $M_3 = M_1^{p\frac{1}{p}} m_2^{(1-p)\frac{-1}{1-p}} = \frac{M_1}{m_2}$ and $h_3 = \frac{M_3}{m_3} = \frac{M_1 M_2}{m_1 m_2} > 1$, so we have the following result as a variation of Theorem E′.

Theorem E. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geqslant A \geqslant m_1 I > 0$ and $M_2 I \geqslant B \geqslant m_2 I > 0$. Put $m = \frac{m_1}{M_2}$, $M = \frac{M_1}{m_2}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Let $p \in (0,1)$. Then the following inequalities hold*:

(i) $(A * I)^p (B * I)^{1-p} \geqslant A^p * B^{1-p} \geqslant K(p)(A * I)^p (B * I)^{1-p}$,

(ii) $(A * I)^p (B * I)^{1-p} \geqslant A^p * B^{1-p} \geqslant (A * I)^p (B * I)^{1-p} - g(p)(B^{1-p} * I)$,

where $g(p) = m^p \left[\frac{h^p - h}{h-1} + (1-p)\left(\frac{h^p - 1}{p(h-1)}\right)^{\frac{p}{p-1}}\right]$ *and $K(p)$ is defined in* (1.2).

We shall investigate several product type and difference type inequalities associated with $A \log A$ by applying the basic property to Theorems A–E which are Kantorovich type inequalities.

2. Several product type and difference type inequalities associated with $A \log A$

In this section we shall state the following several product type and difference type inequalities associated with $A \log A$.

Theorem 2.1. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geqslant A \geqslant m_1 I > 0$ and $M_2 I \geqslant B \geqslant m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and*

$h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. *Let Φ be a normalized positive linear map on $B(H)$. Then the following inequalities hold*:

(i) $$[\log S(1)]\Phi(B) + T(\Phi(A)|\Phi(B)) \geqslant \Phi(T(A|B)) \geqslant T(\Phi(A)|\Phi(B)),$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1)\Phi(A) + T(\Phi(A)|\Phi(B)) \geqslant \Phi(T(A|B)) \geqslant T(\Phi(A)|\Phi(B)),$$

(iii) $$\log S(1)\Phi(A) + \Phi(S(A|B)) \geqslant S(\Phi(A)|\Phi(B)) \geqslant \Phi(S(A|B)),$$

where $S(X|Y)$ and $T(X|Y)$ are defined in (1.8) *and* (1.9) *and $S(1)$ is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.1 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one in [7, Theorem 7].

Corollary 2.2. *Let A be strictly positive operator on a Hilbert space H satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$ and $h = \frac{M}{m} > 1$ and Φ be a normalized positive linear map on $B(H)$. Then the following inequalities hold*:

(i) $$[\log S(1)]\Phi(A) + \Phi(A)\log\Phi(A) \geqslant \Phi(A\log A) \geqslant \Phi(A)\log\Phi(A),$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1) + \Phi(A)\log\Phi(A) \geqslant \Phi(A\log A) \geqslant \Phi(A)\log\Phi(A),$$

(iii) $$\log S(1) + \Phi(\log A) \geqslant \log\Phi(A) \geqslant \Phi(\log A),$$

where $S(1)$ is defined in (1.1).

We remark that the first inequality of (i) in Corollary 2.2 is the reverse inequality of the second one which is known by [1, Theorem 4] and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one which is known by Jensen inequality.

Theorem 2.3. *Let A, B, C and D be strictly positives operators on a Hilbert space H such that $M_1 I \geqslant A \otimes B \geqslant m_1 I > 0$ and $M_2 I \geqslant C \otimes D \geqslant m_2 I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold*:

$$\text{(i)}\quad \begin{aligned}&[\log S(1)](C * D) + T(A * B|C * D)\\ &\quad\geqslant T(A|C) * D + C * T(B|D)\\ &\quad\geqslant T(A * B|C * D),\end{aligned}$$

$$\text{(ii)}\quad \begin{aligned}&\frac{mh\log h}{h-1}(S(1)-1)(A * B) + T(A * B|C * D)\\ &\quad\geqslant T(A|C) * D + C * T(B|D)\\ &\quad\geqslant T(A * B|C * D),\end{aligned}$$

$$\text{(iii)}\quad \begin{aligned}&[\log S(1)](A * B) + S(A|C) * B + A * S(B|D)\\ &\quad\geqslant S(A * B|C * D)\\ &\quad\geqslant S(A|C) * B + A * S(B|D),\end{aligned}$$

where $S(X|Y)$ and $T(X|Y)$ are defined in (1.8) *and* (1.9) *and $S(1)$ is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.3 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one.

Corollary 2.4. *Let A and B be strictly positive operators on a Hilbert space H such that $M_1 I \geqslant A \geqslant m_1 I > 0$ and $M_2 I \geqslant B \geqslant m_2 I > 0$. Put $m = m_1 m_2$, $M = M_1 M_2$ and $h = \frac{M}{m} = \frac{M_1 M_2}{m_1 m_2} > 1$. Then the following inequalities hold*:

$$\text{(i)}\quad \begin{aligned}&[\log S(1)](A * B) + (A * B)\log(A * B)\\ &\quad\geqslant A * (B\log B) + (A\log A) * B\\ &\quad\geqslant (A * B)\log(A * B),\end{aligned}$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1)+(A*B)\log(A*B) \geqslant A*(B\log B)+(A\log A)*B \geqslant (A*B)\log(A*B),$$

(iii) $$\log S(1)+(\log A)*I+I*(\log B) \geqslant \log(A*B) \geqslant (\log A)*I+I*(\log B),$$

where $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Corollary 2.4 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one.

Theorem 2.5. *Let A and B be strictly positive operators on a Hilbert space H such that* $M_1I \geqslant A \geqslant m_1I > 0$ *and* $M_2I \geqslant B \geqslant m_2I > 0$. *Put* $m=\frac{m_1}{M_2}$, $M=\frac{M_1}{m_2}$ *and* $h=\frac{M}{m}=\frac{M_1M_2}{m_1m_2}>1$. *Then the following inequalities hold*:

(i) $$[\log S(1)](A*I)+A*\log B+(A*I)\log(A*I) \geqslant (A\log A)*I+(A*I)\log(B*I) \geqslant A*\log B+(A*I)\log(A*I),$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1)+A*\log B+(A*I)\log(A*I) \geqslant (A\log A)*I+(A*I)\log(B*I) \geqslant A*\log B+(A*I)\log(A*I),$$

(iii) $$[\log S(1)](B*I)+(\log A)*B+(B*I)\log(B*I) \geqslant I*(B\log B)+(\log(A*I))(B*I) \geqslant (\log A)*B+(B*I)\log(B*I),$$

where $S(1)$ *is defined in* (1.1).

We remark that the first inequality of (i) in Theorem 2.5 is the reverse inequality of the second one and also the first inequality of (ii) is the reverse inequality of the second one, and the first inequality of (iii) is the reverse inequality of the second one.

3. Propositions to prove the results in Section 2

We prepare the following propositions to prove the results in Section 2.

Proposition 3.1. *Let Φ be a normalized positive linear map on $B(H)$. Then*

$$\frac{\mathrm{d}\Phi(f(p))}{\mathrm{d}p} = \Phi\left(\frac{\mathrm{d}f(p)}{\mathrm{d}p}\right) \quad \textit{holds for any real number } p, \tag{3.1}$$

where $f(p)$ is a differentiable function of real number p. In particular

$$\frac{\mathrm{d}\Phi(A^p)}{\mathrm{d}p} = \Phi\left(\frac{\mathrm{d}A^p}{\mathrm{d}p}\right) = \Phi(A^p \log A) \quad \textit{holds for any real number } p \neq 0. \tag{3.2}$$

Proof. As Φ is a normalized positive linear map on $B(H)$, we have

$$\begin{aligned}\frac{\mathrm{d}\Phi(f(p))}{\mathrm{d}p} &= \lim_{\Delta p\to 0} \frac{\Phi(f(p+\Delta p)) - \Phi(f(p))}{\Delta p}\\ &= \lim_{\Delta p\to 0} \Phi\left(\frac{f(p+\Delta p) - f(\Delta p)}{\Delta p}\right) \quad \text{by linearity of } \Phi\\ &= \Phi\left(\frac{\mathrm{d}f(p)}{\mathrm{d}p}\right). \qquad \square\end{aligned}$$

Proposition 3.2

(i) $\lim_{p\to 0}(X\sharp_p Y)' = X^{\frac{1}{2}} \log\left[\left(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}}\right)\right] X^{\frac{1}{2}} = S(X|Y)$ *for $X > 0$ and $Y > 0$ where $S(X|Y)$ is defined in* (1.8),

(ii) $\lim_{p\to 1}(X\sharp_p Y)' = (X\sharp Y)X^{-1}S(X|Y)X^{-1}(X\sharp Y) = T(X|Y)$ *for $X > 0$ and $Y > 0$ where $T(X|Y)$ is defined in* (1.9),

(iii) $S(I|Y) = \log Y$ *and* $T(I|Y) = Y \log Y$ *for* $Y > 0$,

(iv) $S(X|I) = -X \log X$ *and* $T(X|I) = -\log X$ *for* $X > 0$,

(v) $(X(p) * Y(p))' = X'(p) * Y(p) + X(p) * Y'(p)$ *where $X(p)$ and $Y(p)$ are operator functions of real number p.*

Proof

(i)
$$\begin{aligned}\lim_{p\to 0}(X\sharp_p Y)' &= \lim_{p\to 0}\left[X^{\frac{1}{2}}\left(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}}\right)^p X^{\frac{1}{2}}\right]'\\ &= \lim_{p\to 0}\left[X^{\frac{1}{2}}\left(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}}\right)^p \log\left[\left(X^{\frac{-1}{2}} Y X^{\frac{-1}{2}}\right)\right] X^{\frac{1}{2}}\right]\\ &= S(X|Y)\end{aligned}$$

and (i) is shown in [6], we cite its proof for the sake of convenience.

(ii)
$$\begin{aligned}\lim_{p\to 1}(X\sharp_p Y)' &= \lim_{p\to 1}\left[X^{\frac{1}{2}}\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)^p X^{\frac{1}{2}}\right]' \\ &= \lim_{p\to 1}\left[X^{\frac{1}{2}}\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)^p \log\left[\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)\right]X^{\frac{1}{2}}\right] \\ &= \left[X^{\frac{1}{2}}\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)^{\frac{1}{2}}\log\left[\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)\right]\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)^{\frac{1}{2}}X^{\frac{1}{2}}\right] \\ &= \left[X^{\frac{1}{2}}\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)^{\frac{1}{2}}X^{\frac{1}{2}}X^{-1}X^{\frac{1}{2}}\right. \\ &\quad\left.\times\log\left[\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)\right]X^{\frac{1}{2}}X^{-1}X^{\frac{1}{2}}\left(X^{\frac{-1}{2}}YX^{\frac{-1}{2}}\right)^{\frac{1}{2}}X^{\frac{1}{2}}\right] \\ &= (X\sharp Y)X^{-1}S(X|Y)X^{-1}(X\sharp Y)\quad\text{by (1.8) and (1.9).}\end{aligned}$$

(iii) and (iv) are immediate consequence of (i) and (ii).
(v) If U is the isometry of H into $H\otimes H$ such that $Ue_n = e_n\otimes e_n$, where e_n is fixed normal basis of H, then the Hadamard product $A*B$ of operators A and B on H is expressed in [5, Theorem 1] as follows:

$$A*B = U^*(A\otimes B)U. \tag{3.3}$$

Then we have

$$\begin{aligned}(X(p)*Y(p))' &= U^*(X(p)\otimes Y(p))'U\quad\text{by (3.3)} \\ &= U^*(X(p)'\otimes Y(p))U + U^*(X(p)\otimes Y(p)')U \\ &= X'(p)*Y(p) + X(p)*Y'(p)\quad\text{by (3.3).}\qquad\square\end{aligned}$$

Proposition 3.3. *Let $h>1$ and let $f(p)$ be defined by*:

$$f(p) = \frac{h^p-h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\qquad \textit{for } p\in[0,1].$$

Then the following (i)–(v) *hold.*

(i) $f(0) = \lim_{p\to 0} f(p) = 0$,
(ii) $f(1) = \lim_{p\to 1} f(p) = 0$,
(iii) $f(p) = \frac{h^p-h}{h-1}\left(1-K(p)^{\frac{1}{p-1}}\right) \geqslant 0$ *for all $p\in[0,1]$*,
(iv) $f'(0) = \lim_{p\to 0} f'(p) = \log S(1)$,
(v) $f'(1) = \lim_{p\to 1} f'(p) = \frac{-h\log h}{h-1}(S(1)-1)$.

Proof
(i) and (ii) are obvious by L'Hospital theorem.

(iii)
$$
\begin{aligned}
f(p) &= \frac{h^p - h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}} \\
&= \frac{h^p - h}{h-1}\left(1 + (1-p)\left(\frac{h^p-h}{h-1}\right)^{-1}\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\right) \\
&= \frac{h^p - h}{h-1}\left[1 - \left(\frac{1}{(h-1)}\left(\frac{h^p-h}{p-1}\right)^{1-p}\left(\frac{h^p-1}{p}\right)^{p}\right)^{\frac{1}{p-1}}\right] \\
&= \frac{h^p - h}{h-1}\left(1 - K(p)^{\frac{1}{p-1}}\right) \geqslant 0
\end{aligned}
$$

and the inequality holds since $\frac{h^p-h}{h-1} \leqslant 0$ and $1 - K(p)^{\frac{1}{p-1}} \leqslant 0$ for $p \in [0, 1]$ by $K(p) \leqslant 1$ for $p \in [0, 1]$ [14, Theorem 1].

(iv)
$$
\begin{aligned}
\lim_{p\to 0} f'(p) &= \lim_{p\to 0} \frac{h^p \log h}{h-1}\left(1 - K(p)^{\frac{1}{p-1}}\right) \\
&\quad + \lim_{p\to 0} \frac{h^p - h}{h-1}\left(-K(p)^{\frac{1}{p-1}}\right)' \quad \text{by (iii)} \\
&= \frac{\log h}{h-1}\left(1 - K(0)^{\frac{1}{-1}}\right) + \lim_{p\to 0}\frac{1-h}{h-1}(-1)\left(K(p)^{\frac{1}{p-1}}\right)' \\
&= 0 - K'(0) \quad \text{by using (1.5) of basic property} \\
&= \log S(1),
\end{aligned}
$$

(v)
$$
\begin{aligned}
\lim_{p\to 1} f'(p) &= \lim_{p\to 1} \frac{h^p \log h}{h-1}\left(1 - K(p)^{\frac{1}{p-1}}\right) \\
&\quad + \lim_{p\to 1} \frac{h^p - h}{h-1}\left(-K(p)^{\frac{1}{p-1}}\right)' \quad \text{by (iii)} \\
&= \frac{h\log h}{h-1}(1 - S(1)) + 0 \quad \text{by (1.5) of basic property} \\
&= \frac{-h\log h}{h-1}(S(1) - 1). \qquad \square
\end{aligned}
$$

Proposition 3.4. *Let $h \geqslant 1$ and $m > 0$. Let $g(p)$ be defined by*:

$$
g(p) = m^p\left(\frac{h^p-h}{h-1} + (1-p)\left(\frac{h^p-1}{p(h-1)}\right)^{\frac{p}{p-1}}\right) \quad \textit{for } p \in [0, 1].
$$

Then the following (i)–(v) *hold*:

(i) $g(0) = \lim_{p\to 0} g(p) = 0$,

(ii) $g(1) = \lim_{p\to 1} g(p) = 0,$
(iii) $g(p) \geqslant 0$ *for all* $p \in [0, 1]$,
(iv) $g'(0) = \lim_{p\to 0} g'(p) = \log S(1),$
(v) $g'(1) = \lim_{p\to 1} g'(p) = \frac{-mh\log h}{h-1}(S(1) - 1).$

Proof. As $g(p) = m^p f(p)$, where $f(p)$ is the same as in Proposition 3.3.

(i), (ii) and (iii) are obvious by (i), (ii) and (iii) of Proposition 3.3 respectively.

(iv)
$$\begin{aligned}\lim_{p\to 0} g'(p) &= \lim_{p\to 0}(m^p(\log m) f(p) + m^p f'(p))\\ &= f(0)\log m + f'(0)\\ &= \log S(1) \quad \text{by (i) and (iv) of Proposition 3.3.}\end{aligned}$$

(v)
$$\begin{aligned}\lim_{p\to 1} g'(p) &= \lim_{p\to 1}(m^p(\log m) f(p) + m^p f'(p))\\ &= mf(1)\log m + mf'(1)\\ &= 0 + m\frac{-h\log h}{h-1}(S(1) - 1) \quad \text{by (ii) and (v) of Proposition 3.3}\\ &= \frac{-mh\log h}{h-1}(S(1) - 1). \qquad \square\end{aligned}$$

4. Proofs of the results in Section 2

For simplicity, $F'(1)$ means $F'(1-0)$ and $F'(0)$ means $F'(+0)$ and so on.

Proof of Theorem 2.1. Applying basic property to Theorem D, we shall show Theorem 2.1.

Recall the following (4.1) for $S > 0$ and $T > 0$

$$S\sharp_1 T = T \text{ and } S\sharp_0 T = S, \tag{4.1}$$

since $S\sharp_p T$ is defined by $S\sharp_p T = S^{\frac{1}{2}}(S^{\frac{-1}{2}} T S^{\frac{-1}{2}})^p S^{\frac{1}{2}}$ for any $p \in [0, 1]$.

Define $F(p)$ and $G(p)$ by as follows:

$$F(p) = \Phi(A)\sharp_p\Phi(B) - \Phi(A\sharp_p B)$$

and

$$G(p) = \Phi(A\sharp_p B) - K(p)\Phi(A)\sharp_p\Phi(B).$$

Recall the following (4.2) by (3.1) of Proposition 3.1

$$[\Phi(A\sharp_p B)]' = \Phi[(A\sharp_p B)']. \tag{4.2}$$

(i) As $F(1) = \Phi(A)\sharp_1\Phi(B) - \Phi(A\sharp_1 B) = \Phi(B) - \Phi(B) = 0$ by (4.1) and $F(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem D, so $F'(1) \leqslant 0$, that is,

$$\begin{aligned}0 \geqslant F'(1) &= [\Phi(A)\natural_p \Phi(B)]'_{p=1} - [\Phi(A\natural_p B)]'_{p=1}\\ &= [\Phi(A)\natural_p \Phi(B)]'_{p=1} - \Phi[(A\natural_p B)'_{p=1}] \quad \text{by (4.2)}\\ &= T(\Phi(A)|\Phi(B)) - \Phi(T(A|B))\end{aligned}$$

by (ii) of Proposition 3.2 and we have the second inequality.

As $G(1) = \Phi(A\natural_1 B) - K(1)\Phi(A)\natural_1 \Phi(B) = \Phi(B) - K(1)\Phi(B) = 0$ by (4.1) and $K(1) = 1$ by (1.4), and $G(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem D, so $G'(1) \leqslant 0$, that is,

$$\begin{aligned}0 \geqslant G'(1) &= [\Phi(A\natural_p B)]'_{p=1} - K'(1)[\Phi(A)\natural_p \Phi(B)]_{p=1}\\ &\quad - K(1)[\Phi(A)\natural_p \Phi(B)]'_{p=1}\\ &= \Phi((A\natural_p B)'_{p=1}) - K'(1)[\Phi(A)\natural_p \Phi(B)]_{p=1}\\ &\quad - K(1)[\Phi(A)\natural_p \Phi(B)]'_{p=1} \quad \text{by (4.2)}\\ &= \Phi(T(A|B)) - \log S(1)[\Phi(A)\natural_1 \Phi(B)] - T(\Phi(A)|\Phi(B))\\ &= \Phi(T(A|B)) - \log S(1)\Phi(B) - T(\Phi(A)|\Phi(B))\end{aligned}$$

by (ii) of Proposition 3.2, $K(1) = 1$ by (1.4), (4.1) and basic property (1.3), and we have the first inequality.

(ii) We have only to show the first inequality since the second one is shown in (i). Define $H(p)$ as follows

$$H(p) = \Phi(A\natural_p B) - \Phi(A)\natural_p \Phi(B) + g(p)\Phi(A).$$

As $H(1) = \Phi(A\natural_1 B) - \Phi(A)\natural_1 \Phi(B) + g(1)\Phi(A) = \Phi(B) - \Phi(B) + g(1)\Phi(A) = 0$ since $g(1) = 0$ by (ii) of Proposition 3.4 and $H(p) \geqslant 0$ for all $p \in (0, 1)$ by (ii) of Theorem D,

$$\begin{aligned}0 \geqslant H'(1) &= [\Phi(A\natural_p B)]'_{p=1} - [\Phi(A)\natural_p \Phi(B)]'_{p=1} + g'(p)_{p=1}\Phi(A)\\ &= \Phi((A\natural_p B)'_{p=1}) - [\Phi(A)\natural_p \Phi(B)]'_{p=1} + g'(1)\Phi(A) \quad \text{by (4.2)}\\ &= \Phi(T(A|B)) - T(\Phi(A)|\Phi(B)) + g'(1)\Phi(A)\end{aligned}$$

by (ii) of Proposition 3.2 and we have the desired inequality by (v) of Proposition 3.4

$$\frac{mh\log h}{h-1}(S(1)-1)\Phi(A) + T(\Phi(A)|\Phi(B)) \geqslant \Phi(T(A|B)).$$

(iii) As $F(0) = \Phi(A)\natural_0 \Phi(B) - \Phi(A\natural_0 B) = \Phi(A) - \Phi(A) = 0$ by (4.1) and $F(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem D, so $F'(0) \geqslant 0$, that is,

$$\begin{aligned}0 \leqslant F'(0) &= [\Phi(A)\natural_p \Phi(B)]'_{p=0} - [\Phi(A\natural_p B)]'_{p=0}\\ &= [\Phi(A)\natural_p \Phi(B)]'_{p=0} - \Phi\big((A\natural_p B)'_{p=0}\big) \quad \text{by (4.2)}\\ &= S(\Phi(A)|\Phi(B)) - \Phi(S(A|B))\end{aligned}$$

by (i) of Proposition 3.2 and we have second inequality. Next we have

$$G(0) = \Phi(A\sharp_0 B) - K(0)\Phi(A)\sharp_0\Phi(B) = \Phi(A) - K(0)\Phi(A) = 0$$

by (4.1) and $K(0) = 1$ by (1.4) and $G(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem D, so $G'(0) \geqslant 0$, that is,

$$\begin{aligned}
0 \leqslant G'(0) &= [\Phi(A\sharp_p B)]'_{p=0} - K'(0)[\Phi(A)\sharp_p\Phi(B)]_{p=0} \\
&\quad - K(0)[\Phi(A)\sharp_p\Phi(B)]'_{p=0} \\
&= \Phi((A\sharp_p B)'_{p=0}) - K'(0)[\Phi(A)\sharp_p\Phi(B)]_{p=0} \\
&\quad - K(0)[\Phi(A)\sharp_p\Phi(B)]'_{p=0} \quad \text{by (4.2)} \\
&= \Phi(S(A|B)) + \log S(1)[\Phi(A)\sharp_0\Phi(B)] - S(\Phi(A)|\Phi(B)) \\
&= \Phi(S(A|B)) + \log S(1)\Phi(A) - S(\Phi(A)|\Phi(B))
\end{aligned}$$

by (i) of Proposition 3.2, $K(0) = 1$ by (1.4), (4.1) and basic property (1.3) and we have the first inequality. □

Proof of Corollary 2.2. Put $A = I$ in Theorem 2.1. Then $\Phi(I) = I$ and (i) of Theorem 2.1 implies the following under the hypotheses of Theorem 2.1

$$\log S(1)\Phi(B) + T(I|\Phi(B)) \geqslant \Phi(T(I|B)) \geqslant T(I|\Phi(B))$$

and this can be rewritten as follows by (iii) of Proposition 3.2

$$\log S(1)\Phi(B) + \Phi(B)\log\Phi(B) \geqslant \Phi(B\log B) \geqslant \Phi(B)\log\Phi(B)$$

so we have (i) of Corollary 2.2 replacing B by A, and (ii) of Corollary 2.2 is easily shown by the same way as (i). Also (iii) of Theorem 2.1 implies the following

$$\log S(1) + \Phi(S(I|B)) \geqslant S(I|\Phi(B)) \geqslant \Phi(S(I|B)),$$

also this can be rewritten as follows by (iii) of Proposition 3.2

$$\log S(1) + \Phi(\log B) \geqslant \log\Phi(B) \geqslant \Phi(\log B)$$

so we have (iii) of Corollary 2.2 replacing B by A.

Remark 4.1. We remark that we can show an easy direct proof of Corollary 2.2 applying basic property to Theorem A.

Proof of Theorem 2.3. Applying basic property to Theorem C, we shall show Theorem 2.3.

Define $f(p)$ and $g(p)$ defined by

$$F(p) = (A * B)\sharp_p(C * D) - (A\sharp_p C) * (B\sharp_p D)$$

and

$$G(p) = (A\sharp_p C) * (B\sharp_p D) - K(p)(A * B)\sharp_p(C * D).$$

(i) We have the following (4.3)

$$\begin{aligned} F(1) &= (A * B)\sharp_1(C * D) - (A\sharp_1 C) * (B\sharp_1 D) \\ &= C * D - C * D = 0 \quad \text{by (4.1).} \end{aligned} \tag{4.3}$$

As $F(1) = 0$ by (4.3) and $F(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem C, so $F'(1) \leqslant 0$, that is,

$$\begin{aligned} 0 \geqslant F'(1) &= [(A * B)\sharp_p(C * D)]'_{p=1} - [(A\sharp_p C) * (B\sharp_p D)]'_{p=1} \\ &= T(A * B)|(C * D) - [(A\sharp_p C)' * (B\sharp_p D)]_{p=1} \\ &\quad - [(A\sharp_p C) * (B\sharp_p D)']_{p=1} \\ &= T(A * B)|(C * D) - T(A|C) * D - C * T(B|D) \end{aligned}$$

by (ii) and (v) of Proposition 3.2 and (4.1) and the second inequality holds.

Also we have

$$G(1) = (A\sharp_1 C) * (B\sharp_1 D) - K(1)(A * B)\sharp_1(C * D) = C * D - C * D = 0, \tag{4.4}$$

since $K(1) = 1$ by (1.4) and (4.1) holds. As $G(1) = 0$ by (4.4) and $G(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem C , so $G'(1) \leqslant 0$, that is,

$$\begin{aligned} 0 \geqslant G'(1) &= [(A\sharp_p C) * (B\sharp_p D)]'_{p=1} - K'(1)(A * B)\sharp_1(C * D) \\ &\quad - K(1)[(A * B)\sharp_p(C * D)]'_{p=1} \\ &= [(A\sharp_p C) * (B\sharp_p D)]'_{p=1} - \log S(1)(C * D) - T(A * B|C * D) \\ &= [(A\sharp_p C)]'_{p=1} * (B\sharp_1 D) + (A\sharp_1 C) * [(B\sharp_p D)]'_{p=1} \\ &\quad - \log S(1)(C * D) - T(A * B|C * D) \\ &= T(A|C) * D + C * T(B|D) - \log S(1)(C * D) \\ &\quad - T(A * B|C * D) \end{aligned}$$

by (ii) and (v) of Proposition 3.2, (4.1) and basic property (1.3), and the first inequality holds.

(ii) We have only to show the first inequality since the second one is shown in (i).

Define $H(p)$ as follows:

$$H(p) = (A\sharp_p C) * (B\sharp_p D) - (A * B)\sharp_p(C * D) + g(p)(A * B),$$

$$\begin{aligned} H(1) &= (A\sharp_1 C) * (B\sharp_1 D) - (A * B)\sharp_1(C * D) + g(1)(A * B) \\ &= C * D - C * D + 0 = 0, \end{aligned} \tag{4.5}$$

since $g(1) = 0$ by (ii) of Proposition 3.4 and (4.1). As $H(1) = 0$ by (4.5) and $H(p) \geqslant 0$ for all $p \in (0, 1)$ by (ii) of Theorem C, so we have

$$\begin{aligned}0 \geqslant H'(1) &= [(A\sharp_p C) * (B\sharp_p D)]'_{p=1} - [(A * B)\sharp_p (C * D)]'_{p=1} \\ &\quad + g'(1)(A * B) \\ &= [(A\sharp_p C)]'_{p=1} * (B\sharp_1 D)] + [(A\sharp_1 C)] * [(B\sharp_p D)]'_{p=1} \\ &\quad - [(A * B)\sharp_p (C * D)]'_{p=1} + g'(1)(A * B) \\ &= T(A|C) * D + C * T((B|D) - T(A * B|C * D)) \\ &\quad + g'(1)(A * B)\end{aligned}$$

by (ii) and (v) of Proposition 3.2 and (4.1), that is, we have the desired inequality by (v) of Proposition 3.4

$$\frac{mh\log h}{h-1}(S(1)-1)(A * B)+T(A * B|C * D) \geqslant T(A|C) * D+C * T(B|D).$$

(iii) We have

$$\begin{aligned}F(0) &= (A * B)\sharp_0 (C * D) - (A\sharp_0 C) * (B\sharp_0 D) \\ &= A * B - A * B = 0 \quad \text{by (4.1).}\end{aligned} \tag{4.6}$$

As $F(0) = 0$ by (4.6) and $F(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem C, so $F'(0) \geqslant 0$, that is,

$$\begin{aligned}0 \leqslant F'(0) &= [(A * B)\sharp_p (C * D)]'_{p=0} - [(A\sharp_p C) * (B\sharp_p D)]'_{p=0} \\ &= S(A * B|C * D) - [(A\sharp_p C)' * (B\sharp_p D)]_{p=0} \\ &\quad - [(A\sharp_p C) * (B\sharp_p D)']_{p=0} \\ &= S(A * B|C * D) - S(A|C) * B - A * S(B|D)\end{aligned}$$

by (i) and (v) of Proposition 3.2 and (4.1) and the second inequality holds.

Also we have

$$G(0) = (A\sharp_0 C) * (B\sharp_0 D) - K(0)(A * B)\sharp_0 (C * D) = A * B - A * B = 0, \tag{4.7}$$

since $K(0) = 1$ by (1.4) and (4.1). As $G(0) = 0$ by (4.7) and $G(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem C, so $G'(0) \geqslant 0$, that is,

$$\begin{aligned}0 \leqslant G'(0) &= [(A\sharp_p C) * (B\sharp_p D)]'_{p=0} - K'(0)(A * B)\sharp_0 (C * D) \\ &\quad - K(0)[(A * B)\sharp_p (C * D)]'_{p=0} \\ &= [(A\sharp_p C) * (B\sharp_p D)]'_{p=0} + \log S(1)(A * B) - S(A * B|C * D) \\ &= [(A\sharp_p C)]'_{p=0} * (B\sharp_0 D) + (A\sharp_0 C) * [(B\sharp_p D)]'_{p=0} \\ &\quad + \log S(1)(A * B) - S(A * B|C * D) \\ &= S(A|C) * B+A * S(B|D)+\log S(1)(A * B)-S(A * B|C * D).\end{aligned}$$

By (i) and (v) of Proposition 3.2, (4.1) and basic property (1.3), and the first inequality holds. □

Proof of Corollary 2.4. Put $A = B = I$ in Theorem 2.3. Then (i) of Theorem 2.3 implies the following under the hypotheses of Theorem 2.3

$$\begin{aligned} &\log S(1)(C * D) + T(I * I|C * D) \\ &\quad \geqslant T(I|C) * D + C * T(I|D) \\ &\quad \geqslant T(I * I|C * D) \end{aligned}$$

and this can be rewritten as follows by (iii) of Proposition 3.2

$$\begin{aligned} &\log S(1)(C * D) + (C * D)\log(C * D) \\ &\quad \geqslant (C \log C) * D + C * (D \log D) \\ &\quad \geqslant (C * D)\log(C * D) \end{aligned}$$

so we have (i) of Corollary 2.4 replacing C and D by A and B, and (ii) of Corollary 2.4 is easily shown by the same way as (i). Also (iii) of Theorem 2.3 implies the following

$$\begin{aligned} &\log S(1)(I * I) + S(I|C) * I + I * S(I|D) \\ &\quad \geqslant S(I * I|C * D) \\ &\quad \geqslant S(I|C) * I + I * S(I|D), \end{aligned}$$

also this can be rewritten as follows by (iii) of Proposition 3.2

$$\begin{aligned} &\log S(1) + (\log C) * I + I * (\log D) \\ &\quad \geqslant \log(C * D) \\ &\quad \geqslant (\log C) * I + I * (\log D) \end{aligned}$$

so we have (iii) of Corollary 2.4 replacing C and D by A and B. □

Remark 4.2. We remark that we can show an easy direct proof of Corollary 2.4 applying basic property to Theorem B.

Proof of Theorem 2.5. Applying basic property to Theorem E, we shall show Theorem 2.5.

Define $F(p)$ and $G(p)$ by as follows:

$$F(p) = (A * I)^p (B * I)^{1-p} - A^p * B^{1-p}$$

and

$$G(p) = A^p * B^{1-p} - K(p)(A * I)^p (B * I)^{1-p}.$$

Recall the following by (v) of Proposition 3.2

$$F'(p) = (A*I)^p[\log(A*I)](B*I)^{1-p} - (A*I)^p(B*I)^{1-p}\log(B*I)$$
$$-(A^p\log A)*B^{1-p} + A^p*(B^{1-p}\log B)$$

and

$$G'(p) = (A^p\log A)*B^{1-p} - A^p*(B^{1-p}\log B)$$
$$-K'(p)(A*I)^p(B*I)^{1-p}$$
$$-K(p)(A*I)^p[\log(A*I)](B*I)^{1-p}$$
$$+K(p)(A*I)^p(B*I)^{1-p}\log(B*I).$$

(i) As $F(1) = 0$ and $F(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem E, so $F'(1) \leqslant 0$, that is,

$$0 \geqslant F'(1) = (A*I)\log(A*I) - (A*I)\log(B*I)$$
$$-(A\log A)*I + A*(\log B)$$

and the second inequality holds. On the other hand, As $G(1) = 0$ since $K(1) = 1$ by (1.4) and $G(p) \geqslant 0$ for all $p \in (0, 1)$ by (i) of Theorem E, so $G'(1) \leqslant 0$, that is,

$$0 \geqslant G'(1) = (A\log A)*I - A*\log B - K'(1)(A*I)$$
$$-K(1)(A*I)[\log(A*I)] + K(1)(A*I)\log(B*I)$$

and the first inequality holds since $K'(1) = \log S(1)$ by (1.3) and $K(1) = 1$ by (1.4), so we have (i).
(ii) We have only to show the first inequality of (ii) since the second one is shown in (i). Define $H(p)$ as follows

$$H(p) = A^p*B^{1-p} - (A*I)^p(B*I)^{1-p} + g(p)(B^{1-p}*I).$$

Recall the following by (v) of Proposition 3.2

$$H'(p) = (A^p\log A)*B^{1-p} - A^p*(B^{1-p}\log B)$$
$$-(A*I)^p[\log(A*I)](B*I)^{1-p}$$
$$+(A*I)^p(B*I)^{1-p}\log(B*I)$$
$$+g'(p)(B^{1-p}*I) - g(p)((B^{1-p}\log B)*I).$$

As $H(1) = 0$ since $g(1) = 0$ by (ii) of Proposition 3.4 and $H(p) \geqslant 0$ for all $p \in (0, 1)$ in (ii) of Theorem E, so

$$0 \geqslant H'(1) = (A\log A)*I - A*\log B - (A*I)[\log(A*I)]$$
$$+(A*I)\log(B*I) + g'(1)(I*I) - g(1)((\log B)*I),$$

that is, we have the desired result by (v) of Proposition 3.4 since $g(1) = 0$ by (ii) of Proposition 3.4

$$\frac{mh\log h}{h-1}(S(1)-1)+A*\log B+(A*I)\log(A*I)$$
$$\geqslant (A\log A)*I+(A*I)\log(B*I).$$

(iii) As $F(0)=0$ and $F(p)\geqslant 0$ for all $p\in(0,1)$ by (i) of Theorem E, so $F'(0)\geqslant 0$, that is,

$$\begin{aligned}0\leqslant F'(0)=&[\log(A*I)](B*I)-(B*I)\log(B*I)\\&-(\log A)*B+I*(B\log B)\end{aligned}$$

and the second inequality holds. On the other hand, As $G(0)=0$ and $G(p)\geqslant 0$ for all $p\in(0,1)$ by (i) of Theorem E, so $G'(0)\geqslant 0$, that is,

$$\begin{aligned}0\leqslant G'(0)=&(\log A)*B-I*(B\log B)-K'(0)(B*I)\\&-K(0)[\log(A*I)](B*I)+K(0)(B*I)\log(B*I)\end{aligned}$$

and the first inequality holds since $-K'(0)=\log S(1)$ by basic property (1.3) and $K(0)=1$ by (1.4), so we have (iii). Whence the proof is complete. □

5. Parallel results to Section 2 and related remarks

We state an extension of Kantorovich inequality.

Theorem F. *Let A be strictly positive operator satisfying $MI\geqslant A\geqslant mI>0$, where $M>m>0$. Put $h=\frac{M}{m}>1$. Then the following inequalities* (i)–(iii) *hold for every unit vector x and follow from each other*:

(i) $K(h,p)(Ax,x)^p\geqslant(A^px,x)\geqslant(Ax,x)^p$ *for any* $p>1$.
(ii) $(Ax,x)^p\geqslant(A^px,x)\geqslant K(h,p)(Ax,x)^p$ *for any* $1>p>0$.
(iii) $K(h,p)(Ax,x)^p\geqslant(A^px,x)\geqslant(Ax,x)^p$ *for any* $p<0$.

We remark that the latter half inequality in (i) or (iii) of Theorem F and the former half one of (ii) are called *Hölder–McCarthy inequality* and the former one of (i) or (iii) and the latter half one of (ii) can be considered as *generalized Kantorovich inequality* and the reverse inequalities to *Hölder–McCarthy inequality*. (i) and (iii) are in [11] and the equivalence relation among (i)–(iii) is shown in [14, Theorem 3] and several extensions of Theorem F are shown, for example, [17, Theorem 3.2].

Related results to Theorem F and operator inequalities associated with Kantorovich type inequalities are in Chapter III of [12].

In this section we sum up the following results which are obtained as applications of basic property and they are parallel results to Sections 1 and 2.

Theorem G ([13]). *Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following inequalities hold for every unit vector x*:

(i) $$[\log S(1)](Ax,x) + (Ax,x)\log(Ax,x) \geqslant ((A\log A)x,x) \geqslant (Ax,x)\log(Ax,x).$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1) + (Ax,x)\log(Ax,x) \geqslant ((A\log A)x,x) \geqslant (Ax,x)\log(Ax,x).$$

(iii) $$[\log S(1)] + ((\log A)x,x) \geqslant \log(Ax,x) \geqslant ((\log A)x,x).$$

Theorem H ([15]). *Let A_j be strictly positive operator satisfying $MI \geqslant A_j \geqslant mI > 0$ for $j = 1, 2, \ldots, n$, where $M > m > 0$ and $h = \frac{M}{m} > 1$. Also $\lambda_1, \lambda_2, \ldots, \lambda_n$ be any positive numbers such that $\sum_{j=1}^{n}\lambda_j = 1$. Then the following inequalities hold*:

(i) $$[\log S(1)]\sum_{j=1}^{n}\lambda_j A_j + \left(\sum_{j=1}^{n}\lambda_j A_j\right)\log\left(\sum_{j=1}^{n}\lambda_j A_j\right) \geqslant \sum_{j=1}^{n}\lambda_j A_j\log A_j \geqslant \left(\sum_{j=1}^{n}\lambda_j A_j\right)\log\left(\sum_{j=1}^{n}\lambda_j A_j\right).$$

(ii) $$\frac{mh\log h}{h-1}(S(1)-1) + \left(\sum_{j=1}^{n}\lambda_j A_j\right)\log\left(\sum_{j=1}^{n}\lambda_j A_j\right) \geqslant \sum_{j=1}^{n}\lambda_j A_j\log A_j \geqslant \left(\sum_{j=1}^{n}\lambda_j A_j\right)\log\left(\sum_{j=1}^{n}\lambda_j A_j\right).$$

(iii) $$[\log S(1)] + \sum_{j=1}^{k}\lambda_j\log A_j \geqslant \log\left(\sum_{j=1}^{k}\lambda_j A_j\right) \geqslant \sum_{j=1}^{k}\lambda_j\log A_j.$$

We remark (iii) for $n = 2$ of Theorem H is shown in [9].

The following interesting result is shown in [6].

Theorem I. *Let A be strictly positive operator satisfying $MI \geqslant A \geqslant mI > 0$. Also let $h = \frac{M}{m} > 1$. Then the following inequality holds for every unit vector x:*

$$S(1)\Delta_x(A) \geqslant (Ax, x) \geqslant \Delta_x(A).$$

where $\Delta_x(A)$ for strictly positive operator A at a unit vector x is defined by

$$\Delta_x(A) = \exp\langle((\log A)x, x)\rangle.$$

$\Delta_x(A)$ is defined in [8]. We remark that (ii) of Theorem F implies Theorem I via basic property. In fact (ii) of Theorem F ensures

$$(Ax, x) \geqslant (A^p x, x)^{\frac{1}{p}} \geqslant K(h, p)^{\frac{1}{p}}(Ax, x) \quad \text{for any } 1 > p > 0. \tag{5.1}$$

and is easily verified that $\lim_{p\to 0}(A^p x, x)^{\frac{1}{p}} = \Delta_x(A)$ and $\lim_{p\to 0} K(h, p)^{\frac{1}{p}} = \frac{1}{S(1)}$ by (1.5), so that (5.1) implies Theorem I.

Interesting closely related results to Theorems G and H are in [24].

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